

# On the description of Lax type integrable dynamical systems within the Marsden-Weinstein reduction method and its relationship with the AKS-BK and $R$ -matrix approaches

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## Аннотация

An algebraic approach to constructing nonlinear Lax type integrable dynamical systems, based on the Marsden-Weinstein reduction method on canonically symplectic manifolds with group symmetry, is proposed. Its natural relationship with the well known Adler-Kostant-Souriau-Berezin-Kirillov method and the associated  $R$ -matrix method is analyzed.

## 1 Introduction

As it is well known [2, 1, 13, 8], the most popular canonically manifolds are supplied by cotangent spaces  $M := T^*(P)$  to some "coordinates" phase spaces  $P$ , which can often possess additional symmetry properties. If this symmetry can be identified with some Lie group  $\tilde{G}$  action on the phase space  $P$  and its natural extension on the whole manifold  $M$  proves to be symplectic and even more, Hamiltonian, the Marsden-Wienstein reduction method [1, 6] makes it possible to construct new Hamiltonian flows on the smaller invariant reduced phase space  $\bar{M}_\xi := M_\xi/G_\xi$  subject to the group invariant constraint  $p := \xi \in \mathcal{G}^*$  for some specially chosen element  $\xi \in \mathcal{G}^*$ , where  $p : M \rightarrow \mathcal{G}^*$  is the related momentum mapping on the symplectic manifold  $M$  and  $\mathcal{G}^*$  is the adjoint space to the Lie algebra  $\mathcal{G}$  of the group  $\text{Lie } \tilde{G}$ .

As the corresponding Hamiltonian flows on the reduced phase space  $\bar{M}_\xi$  possesses very often very interesting properties important for applications in many branches of mathematics and physics, their studies were topics of many researches during the past decades. Being interested in mathematical properties of the Lax type integrable flows, we observed that their modern Lie algebraic description by means of the Hamiltonian group action classical Lie-Poisson-Adler-Kostant-Berezin-Kirillov scheme on the adjoint space  $\hat{\mathcal{G}}^*$  to the Lie algebra  $\hat{\mathcal{G}}$  of a suitably chosen group  $G$  is a natural consequence of applying the mentioned above Marsden-Weinstein reduction method to canonical symplectic phase space  $M = T^*(P)$  with the basis space  $P$ , to be a specially chosen Lie algebra  $\mathcal{G}$  with the naturally related Hamiltonian group  $\tilde{G}$  action on the symplectic phase space  $M$ . Moreover, such classical integrability theory ingredients as the  $R$ -structures [16] and the related commutation properties of the related transfer matrices are also naturally retrieved from the Marsden-Weinstein reduction method within the scheme specified above. These and some related aspects of this reduction technique are topics of this investigation.

## 2 Loop group, canonically symplectic manifold and Hamiltonian action

There is considered a complex matrix Lie group  $G = SL(\nu; \mathbb{C})$ ,  $\nu \in \mathbb{Z}_+$ , its Lie algebra  $\mathcal{G}$ , and a related [8, 9, 13] formal loop group  $\tilde{G} \subset C^\infty(\mathbb{S}^1; \text{Hol}(\mathbb{C}; G))$  of  $G$ -valued functions on the circle  $\mathbb{S}^1$ , meromorphically depending on the complex parameter  $\lambda \in \mathbb{C}$ . Its Lie algebra  $\tilde{\mathcal{G}}$  can be viewed as

the completion

$$\tilde{\mathcal{G}} = \bigcup_{n \in \mathbb{Z}} \left\{ \sum_{j=-\infty}^n \tilde{X}_j \lambda^j : \tilde{X}_j \in C^\infty(\mathbb{S}^1; \mathcal{G}), j = \overline{-\infty, n} \right\}. \quad (2.1)$$

By the standard procedure [8, 6] one can construct the centrally extended current algebra  $\hat{\mathcal{G}} := \tilde{\mathcal{G}} \oplus \mathbb{C}$ , on which the adjoint loop group  $\tilde{G}$ -action is defined: for any  $g \in \tilde{G}$

$$g : (T, c) \rightarrow (gTg^{-1}, c + (g^{-1}g_x, T)_{-1}). \quad (2.2)$$

Here  $(T, c) \in \hat{\mathcal{G}}$  and  $(\cdot, \cdot)_{-1} : \tilde{\mathcal{G}} \times \tilde{\mathcal{G}} \rightarrow \mathbb{C}$  is the following nondegenerate symmetric scalar product on  $\tilde{\mathcal{G}}$ :

$$(A, B)_{-1} := \text{res} \int_0^{2\pi} \text{tr}(A(x; \lambda)B(x; \lambda)) = (B, A)_{-1}, \quad (2.3)$$

for any  $A, B \in \tilde{\mathcal{G}}$ . The scalar product (2.3) is ad-invariant, that is

$$(A, [B, C])_{-1} = ([A, B], C)_{-1} \quad (2.4)$$

for any elements  $A, B$  and  $C \in \tilde{\mathcal{G}}$ .

Define now the canonically symplectic phase space  $M := T^*(\hat{\mathcal{G}}) \simeq (\hat{\mathcal{G}}^*, \hat{\mathcal{G}})$  with the corresponding Liouville 1-form on  $M$ :

$$\alpha^{(1)}(T, c; l, k) = (l, dT)_{-1} + kdc, \quad (2.5)$$

whose external derivative gives the symplectic structure on the functional manifold  $M$ :

$$\omega^{(2)}(T, c; l, k) := d\alpha^{(1)}(T, c; l, k) = (dl, \wedge dT)_{-1} + dk \wedge dc. \quad (2.6)$$

Similarly to (2.2) one can naturally extend the group  $\tilde{G}$ -action on the whole phase space  $M$ , having

$$g : (l, k) \rightarrow (glg^{-1} - kg_xg^{-1}, k) \quad (2.7)$$

for any  $(l, k) \in \hat{\mathcal{G}}^*$  and  $g \in \tilde{G}$  as the corresponding co-adjoint action of the current group  $\tilde{G}$  on the adjoint linear space  $\hat{\mathcal{G}}^*$ . The following lemma is almost evident.

**Lemma 2.1** *The  $\tilde{G}$ -group action (2.2) and (2.7) on the symplectic phase space  $M$  is symplectic and Hamiltonian.*

**Proof.** It is easy to check that the canonical Liouville 1-form (2.5) on the manifold  $M$  is  $\tilde{G}$ -invariant:

$$\begin{aligned} g^* \alpha^{(1)}(T, c; l, k) &= (glg^{-1} - kg_xg^{-1}, gdTg^{-1})_{-1} + k(dc + (g^{-1}g_x, dT)_{-1}) = \\ &= (glg^{-1}, gdTg^{-1})_{-1} - k(g_xg^{-1}, gdTg^{-1})_{-1} + kdc + k(g^{-1}g_x, dT)_{-1} = \\ &= (l, g^{-1}gdTg^{-1}g)_{-1} - k(g^{-1}g_xg^{-1}g, dT)_{-1} + kdc + k(g^{-1}g_x, dT)_{-1} = \\ &= (l, dT)_{-1} + kdc = \alpha^{(1)}(T, c; l, k). \end{aligned} \quad (2.8)$$

From (2.8), owing to the expression (2.6), one obtains the symplectic form invariance

$$g^* \omega^{(2)}(T, c; l, k) = \omega^{(2)}(T, c; l, k) \quad (2.9)$$

for any element  $(T, c; l, k) \in M$ .

To state the Hamiltonian  $\tilde{G}$ -action on the symplectic manifold  $M$  we take the group flow  $g(t) := \exp(tX)$  for  $t \in \mathbb{R}$ ,  $X \in \tilde{\mathcal{G}}$ , and find the respectively generated vector field  $K_X : M \rightarrow T(M)$  on the phase space  $M$ :

$$\begin{aligned} K_X(T, c; l, k) &:= \\ &= \left. \frac{d}{dt} (g(t)Tg(t)^{-1}, c + (g(t)^{-1}g_x(t), T)_{-1}; g(t)lg(t)^{-1} - kg_x(t)g(t)^{-1}, k) \right|_{t=0} = \\ &= ([X, T], (X_x, T)_{-1}; [X, l] - kX_x, 0), \end{aligned} \quad (2.10)$$

by a Hamiltonian function  $H_X : M \rightarrow \mathbb{C}$  owing to the canonical relationship  $-dH_X = i_{K_X}\omega^{(2)}$  :

$$\begin{aligned} -dH_X &= -(\partial H/\partial l, dl)_{-1} - (\partial H_X/\partial T, dT)_{-1} + \\ &+ \partial H_X/\partial k \, dk + \partial H_X/\partial c \, dc = \\ &= ([X, l] - kX_x, dT)_{-1} - (dl, [X, T])_{-1} - (X_x, T)_{-1} dk. \end{aligned} \quad (2.11)$$

As a consequence of (2.11) one obtains that

$$\begin{aligned} \partial H_X/\partial l &= [X, T], & \partial H_X/\partial T &= kX_x - [X, l], \\ \partial H_X/\partial k &= (X_x, T)_{-1}, & \partial H_X/\partial c &= 0 \end{aligned} \quad (2.12)$$

for any point  $(T, k; l, c) \in M$ . From (2.12) follows the expression

$$H_X = ([T, l] - kT_x, X)_{-1} := (p(T, c; l, k), X)_{-1}, \quad (2.13)$$

linear with respect to the generator element  $X \in \tilde{\mathcal{G}}$ . It means that the loop group  $\tilde{\mathcal{G}}^*$  action on the symplectic manifold  $M$  is Hamiltonian by definition [1, 12].  $\triangleright$

The corresponding mapping  $p : M \rightarrow \tilde{\mathcal{G}}^*$ , where

$$p(T, c; l, k) = [T, l] - kT_x, \quad (2.14)$$

is called the momentum mapping [1, 6, 12] which can be constrained to be a fixed for further applying to the phase space  $M$  the Marsden-Weinstein reduction procedure [1].

Let us describe in detail the related symplectic structure on the  $\xi$ -level submanifold

$$M_\xi := \left\{ (T, c; l, k) \in M : [T, l] - kT_x = \xi \in \tilde{\mathcal{G}}^* \right\} \quad (2.15)$$

for a fixed element  $\xi \in \tilde{\mathcal{G}}^*$ . As a more natural case we will put, by definition, that  $\xi = 0 \in \tilde{\mathcal{G}}^*$ . The corresponding isotropy group  $\tilde{G}_\xi = \tilde{G}$ , as  $\text{Ad}_g^* \xi|_{\xi=0} = 0$  holds for any element  $g \in \tilde{G}$ . To proceed further we need some additional properties of the submanifold  $M_\xi \subset M$ , which we will describe in the next section

### 3 Marsden-Weinstein reduction, commuting vector fields and Poisson bracket

In this section we will be interested in describing the submanifold  $M_\xi \subset M$  parametrized by the points of the reduced phase space  $\bar{M}_\xi := M_\xi/G_\xi$ . It is known [1, 2], that this parametrization uniquely parametrized the points  $(\bar{T}, \bar{c}; \bar{l}, \bar{k}) \in M_\xi \subset M$ , which are invariant with respect to the appropriate loop group  $\tilde{G}$  action (2.2) and (2.7). The last property make it possible [1, 2, 6, 11] to define on the phase space  $\bar{M}_\xi$  the reduced nondegenerate symplectic structure on the phase space  $\bar{M}_\xi$  by means of the appropriate symplectic structure on the submanifold  $M_\xi$ . Let us consider the point  $(\bar{T}, \bar{c}; \bar{l}, \bar{k}) \in M_\xi$ , where the elements  $\bar{T} \in \tilde{\mathcal{G}}$ ,  $\bar{k} \in \mathbb{C}$ , according to the definition (2.15), satisfy such differential expressions:

$$[\bar{T}, \bar{l}] - \bar{k}\bar{T}_x = 0, \quad \bar{k}_x = 0, \quad (3.1)$$

for all  $x \in \mathbb{S}^1$ . Consider now a Hamiltonian vector field  $-\bar{k}d/d\tau, \tau \in \mathbb{C}$ , on the submanifold  $M_\xi$ , generated by the element  $X = \bar{l} \in \tilde{\mathcal{G}}^*$  owing to the expressions

$$-\bar{k}\bar{T}_\tau = [\bar{l}, \bar{T}] = -[\bar{T}, \bar{l}] = -\bar{k}\bar{T}_x, \quad -\bar{k}\bar{l}_\tau = \bar{k}\bar{l}_x. \quad (3.2)$$

From (3.2) one follows, that the equality  $\frac{d}{d\tau} = \frac{d}{dx}$  holds on the reduced phase space  $\bar{M}_\xi$ . Let us compute additionally the evolution of the element  $\bar{c} \in \mathbb{C}$  with respect to this vector field  $d/d\tau$  on  $\bar{M}_\xi$  :

$$-\bar{k}\bar{c}_\tau = (\bar{l}_x, \bar{T})_{-1} = -(\bar{l}, \bar{T}_x)_{-1} = -(\bar{l}, \bar{k}^{-1}[\bar{T}, \bar{l}])_{-1} = \bar{k}^{-1}([\bar{l}, \bar{l}], \bar{T})_{-1} = 0, \quad (3.3)$$

coinciding with the *a priori* assumed condition  $d\bar{c}/dx = 0$  for any  $x \in \mathbb{S}^1$ .

Define similarly a vector field  $d/dt$ ,  $t \in \mathbb{C}$ , on the reduced phase space  $\bar{M}_\xi$ , generated by the Lie algebra element  $q(\bar{l}) \in \tilde{\mathcal{G}}$ , depending in such a way on the basis element  $\bar{l} \in \tilde{\mathcal{G}}^*$ , that

$$\bar{T}_t = [q(\bar{l}), \bar{T}], \quad \bar{l}_t = [q(\bar{l}), \bar{l}] - \bar{k}\bar{l}_x, \quad \bar{c}_t = (q_x(\bar{l}), \bar{T}), \quad \bar{k}_t = 0. \quad (3.4)$$

The latter, in particular, means that the flows  $d/dt$  and  $d/dx$  on the reduced phase space  $\bar{M}_\xi$  possess the countable set  $\gamma_n(\bar{l}) := \text{tr} \bar{T}^n(\bar{l})$ ,  $n \in \mathbb{Z}$ , of conservation laws, where by definition, the element  $\bar{T}(\bar{l}) \in \tilde{\mathcal{G}}$  satisfies for a given element  $\bar{l} \in \tilde{\mathcal{G}}^*$  the determining equation

$$-\bar{k}\bar{T}_x(\bar{l}) = [\bar{l}, \bar{T}(\bar{l})] \quad (3.5)$$

for all  $x \in \mathbb{S}^1$ . From the equations (3.5) one easily obtains that upon the reduced phase space  $\bar{M}_\xi$

$$\begin{aligned} \bar{c}_t &= (q(\bar{l})_x, \bar{T})_{-1} = \bar{k}^{-1}([q(\bar{l}), \bar{l}] - \bar{l}_t, \bar{T})_{-1} = \\ &= \bar{k}^{-1}([q(\bar{l}), \bar{l}], \bar{T})_{-1} - \bar{k}^{-1}(\bar{l}_t, \bar{T}) = \bar{k}^{-1}([\bar{T}, q(\bar{l})], \bar{l})_{-1} - \\ &\quad - \bar{k}^{-1}(\bar{l}_t, \bar{T})_{-1} = -\bar{k}^{-1}(\bar{l}, \bar{T}_t)_{-1} - \bar{k}^{-1}(\bar{l}_t, \bar{T})_{-1} = \\ &\quad -\bar{k}^{-1} \frac{d}{dt}(\bar{l}, \bar{T})_{-1}. \end{aligned} \quad (3.6)$$

Thus, from the  $t$ -evolution (3.6) of the parameter  $\bar{c} \in \mathbb{C}$  one obtains that the constraint

$$\bar{c} = -\bar{k}^{-1}(\bar{l}, \bar{T})_{-1} \quad (3.7)$$

holds on the reduced phase space  $\bar{M}_\xi$  subject to any vector field  $d/dt$ , generated by the element  $q(\bar{l}) \in \tilde{\mathcal{G}}$ . Moreover, as it is easy to observe, these two vector fields  $d/d\tau$  and  $d/dt$  on the reduced phase space  $\bar{M}_\xi$  are commuting to each other,

$$[d/dt, d/d\tau] = 0. \quad (3.8)$$

The latter is very promising, since the condition (3.8) results in some differential relationships on the components of the reduced matrix  $\bar{l} \in \tilde{\mathcal{G}}^*$ , for which the related linear evolution equation

$$\bar{F}_x = \bar{l}\bar{F}, \quad (3.9)$$

augmented with the compatible differential equation

$$\bar{F}_t = q(\bar{l})\bar{F} \quad (3.10)$$

for matrix  $F \in \tilde{\mathcal{G}}$  are compatible. Threby, these equations (3.9) and (3.10) realize the well known [8, 10, 9, 13, 12, 6] generalized Lax type spectral problem, allowing to integrate the mentioned above differential relationships by means of either the inverse scattering or the spectral transform methods [8, 10, 9, 7] and algebraic geometry methods [10, 9], or other their modern generalizations [13].

To make this aim more constructive, it is necessary to describe the evolution of the vector field  $d/dt$  on the reduced phase space  $\bar{M}_\xi$  in more detail subject to its dependence on the phase space element  $\bar{l} \in \tilde{\mathcal{G}}^*$ . Taking into account that the vector fields  $d/dt$  and  $d/dx$  satisfy the commutation condition (3.8) on the reduced manifold  $M_\xi$ , we will apply the Marsden-Weinstein reduction theory to our symplectic manifold  $M$  with the fixed value of the moment mapping  $\xi = 0$  for computing the basic Poisson bracket

$$\{(\bar{T}, X)_{-1}, (\bar{T}, Y)_{-1}\}_\xi \quad (3.11)$$

of the functions  $(\bar{T}, X)$  and  $(\bar{T}, Y)$  on the reduced phase space  $\bar{M}_\xi$  for arbitrary  $X, Y \in \tilde{\mathcal{G}}^*$ . It can be shown [4, 11, 6] that this Poisson bracket on  $\bar{M}_\xi$  in general equals

$$\{(\bar{T}, X)_{-1}, (\bar{T}, Y)_{-1}\}_\xi = \{(\bar{T}, X)_{-1}, (\bar{T}, Y)_{-1}\}|_{\bar{M}_\xi} - (\xi, [V_X, V_Y])_{-1}|_{\bar{M}_\xi}, \quad (3.12)$$

where, by definition, the mappings  $V_X, V_Y : \bar{M}_\xi \rightarrow \tilde{\mathcal{G}}$  denote the solutions to the following relationships:

$$(\xi, [Z, V_X])_{-1} = K_Z(T, X)_{-1}, \quad (\xi, [Z, V_Y])_{-1} = K_Z(T, Y)_{-1}, \quad (3.13)$$

which hold for all  $Z \in \tilde{\mathcal{G}}$ , and functions  $(\bar{T}, X)_{-1}, (\bar{T}, Y)_{-1} \in \mathcal{D}(\bar{M}_\xi)$  should be extended to those on the whole phase space  $M$  from the related Poisson bracket in such a way that their restrictions upon the submanifold  $M_\xi \subset M$  were  $\tilde{G}$ -invariant.

To apply the Marsden-Weinstein reduction we will take into account that, by definition, there exists a group element  $g(\bar{l}) \in \tilde{G}$  such that for arbitrarily chosen  $l \in \tilde{G}$  the expression

$$l = g(l)\bar{l}g(l)^{-1} - \bar{k}g_x(l)g(l)^{-1} \quad (3.14)$$

holds and satisfying the normalization condition  $g(\bar{l}) = \text{Id} \in \tilde{G}$ . If now to consider the function

$$f_X := (T, g(l)Xg(l)^{-1})_{-1}, \quad (3.15)$$

one can observe that  $f_X|_{\bar{M}_\xi} = (\bar{T}, X)_{-1}$  and, by construction, it is  $\tilde{G}$ -invariant. The latter means that  $f_X \in \mathcal{D}(M_\xi)$  for any  $l \in \tilde{\mathcal{G}}^*$ . Really, for any  $a \in \tilde{G}_\xi = \tilde{G}$

$$\begin{aligned} a \circ f_X &:= (a \cdot T, g(a \circ l)Xg(a \circ l)^{-1})_{-1} = \\ &= (aTa^{-1}, ag(l)Xg(l)^{-1} \cdot a^{-1}) = (T, g(l)Xg(l)^{-1})_{-1} = f_X, \end{aligned} \quad (3.16)$$

where we made use of the property  $g(a \circ l) = a g(l)$ ,  $l \in \tilde{\mathcal{G}}^*$ . The latter holds owing to the definitions (3.14) and (2.7):

$$\begin{aligned} a \circ l &= ala^{-1} - \bar{k}a_xa^{-1} = a(g(l)\bar{l}g(l)^{-1} - \bar{k}g_x(l)g(l)^{-1})a^{-1} - \bar{k}a_xa^{-1} = \\ &= ag(l)\bar{l}(ag(l))^{-1} - \bar{k}ag_x(l)g(l)^{-1}a^{-1} - \bar{k}a_xa^{-1} = \\ &= ag(l)\bar{l}(ag(l))^{-1} - \bar{k}(ag(l))_x(ag(l))^{-1} = \\ &= g(a \circ l)\bar{l}g(a \circ l)^{-1} - \bar{k}g_x(a \circ l)g(a \circ l)^{-1}, \end{aligned} \quad (3.17)$$

giving rise to relationship  $g(a \circ l) = ag(l)$  for any  $a \in \tilde{G}_\xi$  and  $l \in \tilde{\mathcal{G}}^*$ .

Returning back to the Poisson bracket (3.12), we can replace the functions  $(\bar{T}, X)_{-1}$  and  $(\bar{T}, Y)_{-1} \in \mathcal{D}(\bar{M}_\xi)$  with their  $\tilde{G}_\xi$ -invariant extensions  $f_X \in \mathcal{D}(M_\xi)$ . Before calculating the corresponding Poisson bracket

$$\{\bar{f}_X, \bar{f}_Y\}_\xi = \{\bar{f}_X, \bar{f}_Y\}|_{\bar{M}_\xi} - (\xi, [V_X, V_Y])_{-1} = \{f_X, f_Y\}|_{\bar{M}_\xi} - K_{V_X}f_Y|_{\bar{M}_\xi}, \quad (3.18)$$

where  $K_{V_X} : M \rightarrow T(M)$  is the vector field, generated on  $M$  by the element  $V_X \in \tilde{\mathcal{G}}$ , we need to calculate the action  $K_Z f_Y$  for any element  $Z \in \tilde{\mathcal{G}}$ . Similarly to the calculations from [4], one obtains, that on the submanifold  $M_\xi$

$$\begin{aligned} K_Z f_Y &= \frac{d}{d\varepsilon} (\exp(\varepsilon Z)T \exp(-\varepsilon Z), g(\exp(\varepsilon Z) \circ l)Yg(\exp(\varepsilon Z) \circ l)^{-1})_{-1}|_{\varepsilon=0} = \\ &= (T, g(l)[g(l)^{-1}g'(l)([Z, l] - \bar{k}Z_x) - g(l)^{-1}Zg(l), Y]g(l)^{-1})_{-1}. \end{aligned} \quad (3.19)$$

Thus, on the reduced phase space  $\bar{M}_\xi$  the general expression (3.19) entails

$$K_{V_X} f_Y|_{\bar{M}_\xi} = (\bar{T}, [g'(\bar{l}) \cdot ([V_X, \bar{l}] - \bar{k} \frac{d}{dx} V_X) - V_X, Y])_{-1}. \quad (3.20)$$

Thus, the Poisson bracket (3.18), owing to the relationships  $\{f_X, f_Y\} = -\omega^{(2)}(K_{V_X}, K_{V_Y})$  and (3.20), becomes

$$\begin{aligned} \{(\bar{T}, X), (\bar{T}, Y)\}_\xi &= \\ &= (\bar{T}, [g'(\bar{l})(Y), X] + [Y, g'(\bar{l})(X)])_{-1} - \left( \bar{T}, [g'(\bar{l})([V_X, \bar{l}] - \bar{k} \frac{d}{dx} V_X) - V_X, Y] \right)_{-1} = \\ &= (\bar{T}, [g'(\bar{l})(Y), X] + [Y, g'(\bar{l})(X)])_{-1}, \end{aligned} \quad (3.21)$$

where we take into account that owing to (3.13) and (3.20), the expression

$$\left( \bar{T}, [g'(\bar{l})([V_X, \bar{l}] - \bar{k} \frac{d}{dx} V_X) - V_X, Y] \right)_{-1} = K_{V_X} f_Y = (\xi, [K_{V_X}, V_Y])_{-1}|_{\xi=0} = 0.$$

Now one can rewrite the Poisson bracket (3.21) as

$$\{(\bar{T}, X), (\bar{T}, Y)\}_\xi = (\bar{T}, [X, Y]_D)_{-1}, \quad (3.22)$$

where, by definition, we have introduced the classical  $D$ -matrix structure in the Lie algebra  $\tilde{\mathcal{G}}^*$ :

$$[X, Y]_D := [D(X), Y] + [X, D(Y)], \quad (3.23)$$

where  $X, Y \in \tilde{\mathcal{G}}^*$  and the linear homomorphism  $D : \tilde{\mathcal{G}}^* \rightarrow \tilde{\mathcal{G}}^*$  is defined as

$$D(X) := -g'(\bar{l})(X). \quad (3.24)$$

The mapping (3.24) should necessarily satisfy [5] the well known condition

$$(\bar{T}, [X, [D(Y), D(Z)] - D[Y, Z]_D])_{-1} + (\bar{T}, [X, \{(\bar{T}, Y), (\bar{T}, Z)\}]) + \text{cycles} = 0 \quad (3.25)$$

for any  $X, Y \in \tilde{\mathcal{G}}^*$  and  $Z \in \tilde{\mathcal{G}}$ .

Now it is useful to recall that the mapping  $g : \tilde{\mathcal{G}}^* \rightarrow \tilde{\mathcal{G}}$  satisfies the relationship (3.14), which entails [3] the following differential expression

$$[g'(\bar{l})(X), \bar{l}] - \bar{k} \frac{d}{dx} g'(\bar{l})(X) + Q(\bar{l})(X) = X \quad (3.26)$$

for any  $X \in \tilde{\mathcal{G}}^*$ , where  $Q(\bar{l}) : \tilde{\mathcal{G}}^* \rightarrow \tilde{\mathcal{G}}^*$  is a suitable mapping, depending from the chosen reduction  $\mathcal{G}^* \ni l \rightarrow \bar{l}(l) \in \mathcal{G}^*$ .

The mapping (3.24) satisfies still an additional relationship, which can be obtained from the group  $\tilde{G}$ -action on the element  $\bar{T}(\bar{l}) \in \tilde{\mathcal{G}}$ :

$$T(l) = g(l)\bar{T}(\bar{l})g(l)^{-1}, \quad (3.27)$$

following naturally from (3.14). Differentiation of (3.27) with respect to  $l \in \tilde{\mathcal{G}}^*$  in the point  $l = \bar{l}$ , gives rise to the expression

$$T'(\bar{l})(X) = [g'(\bar{l})(X), \bar{T}(\bar{l})] + \Phi(\bar{l})(X) \quad (3.28)$$

for an arbitrary  $X \in \tilde{\mathcal{G}}^*$  and some mapping  $\Phi(\bar{l}) : \tilde{\mathcal{G}}^* \rightarrow \tilde{\mathcal{G}}$ . Moreover, since the matrix (3.27) satisfy the relationship (3.5), its differentiation with respect to  $\bar{l} \in \tilde{\mathcal{G}}^*$  entails the differential expression:

$$\bar{k} \frac{d}{dx} T'(\bar{l})(Y) + [\bar{l}, T'(\bar{l})(Y)] = [\bar{T}(\bar{l}), Y], \quad (3.29)$$

which holds for any  $Y \in \tilde{\mathcal{G}}^*$ . Now we assume that the mapping  $\Phi(\bar{l}) : \tilde{\mathcal{G}}^* \rightarrow \tilde{\mathcal{G}}$  is chosen in such a way that

$$\bar{k} \frac{d}{dx} \Phi(\bar{l})(Y) + [\bar{l}, \Phi(\bar{l})(Y)] = 0 \quad (3.30)$$

for all  $Y \in \tilde{\mathcal{G}}^*$ . Then, substituting (3.28) into the expression (3.29), owing to (3.26) and (3.30), one obtains that for any  $Y \in \tilde{\mathcal{G}}^*$

$$[\bar{Q}(\bar{l})(Y), \bar{T}(\bar{l})] = 0. \quad (3.31)$$

The latter, in particular, means that the mapping  $Q(\bar{l}) : \tilde{\mathcal{G}}^* \rightarrow \tilde{\mathcal{G}}^*$  can be chosen in the equivalent tensor form such that  $Q(\bar{l}) = \bar{T}(\bar{l}) \otimes \bar{Q}(\bar{l})$  satisfies (3.31). The obtained above results can be formulated as the following proposition.

**Proposition 3.1** *The Poisson bracket (3.11) on the reduced phase space  $\bar{M}_\xi$  represented as a  $D$ -structure (3.22) on the linear space  $\tilde{\mathcal{G}}^*$ , naturally generated by the gauge transformation (3.14), which reduces the arbitrary element  $l \in \tilde{\mathcal{G}}^*$  into element  $\bar{l} \in \tilde{\mathcal{G}}^*$ , uniquely defined on  $\bar{M}_\xi$ .*

As a consequence of representation (3.22) we automatically obtains that there exists the infinite hierarchy commuting to each other functionals with respect to the Poisson bracket on the phase space  $\bar{M}_\xi$ . The latter follows from the tensor form of the Poisson bracket (3.11) in the space  $\tilde{\mathcal{G}} \otimes \tilde{\mathcal{G}}$ :

$$\{\bar{T}(\bar{l})(\lambda) \otimes \bar{T}(\bar{l})(\mu)\}_\xi = [D(\lambda, \mu), \bar{T}(\bar{l})(\lambda) \otimes I + I \otimes \bar{T}(\bar{l})(\mu)] \quad (3.32)$$

which holds for arbitrary  $\lambda, \mu \in \mathbb{C}$ . The trace operation in (3.32) nullify the Poisson bracket on the phase space  $\bar{M}_\xi$  for the functionals  $\text{tr} \bar{T}(\bar{l})(\lambda)$  and  $\text{tr} \bar{T}(\bar{l})(\mu)$  for arbitrary  $\lambda, \mu \in \mathbb{C}$ .

## 4 Monodromy matrix, associated $R$ -structure and Lie-Poisson bracket

Proceed now to analyzing possible forms of  $D$ -mapping (3.24) as a function on the reduced phase space  $\bar{M}_\xi$ . Since the parameter  $\bar{k} \in \mathbb{C}$  is constant, its value for convenience will be put by  $\bar{k} = -1$ . Thus, taking into account the definition (3.24), the determining  $D$ -structure equation (3.26) takes the form:

$$[D(\bar{l})(Y), \bar{l}] + \frac{d}{dx} D(\bar{l})(Y) + Y = Q(\bar{l})(Y) \quad (4.1)$$

for any element  $Y \in \tilde{\mathcal{G}}^*$ .

Let us consider the linear matrix equation

$$\bar{F}_x(x, s; \lambda) = \bar{l}(x; \lambda) \bar{F}(x, s; \lambda), \quad (4.2)$$

where  $\bar{l}(x; \lambda) \in \tilde{\mathcal{G}}^*$ ,  $\bar{F} \in \tilde{G}$ , with Cauchy data at a point  $x = s \in \mathbb{S}^1$ :

$$\bar{F}(x, s; \lambda)|_{x=s} = \mathbb{I}. \quad (4.3)$$

The corresponding normalized monodromy matrix

$$\bar{T}(x; \lambda) := \bar{F}(x + 2\pi, x; \lambda) - \nu^{-1} \mathbb{I} \text{tr} \bar{F}(x + 2\pi, x; \lambda), \quad (4.4)$$

for  $x \in \mathbb{S}^1$  and arbitrary  $\lambda \in \mathbb{C}$  satisfies the the differential expression

$$\bar{T}_x - [\bar{T}, \bar{l}] = 0, \quad (4.5)$$

exactly coinciding with (3.5). Thus, if by means of the co-adjoint transformation (2.7) this chosen matrix  $l \in \tilde{\mathcal{G}}^*$  will be transformed into the matrix  $\bar{l} \in \tilde{\mathcal{G}}^*$ , then the corresponding monodromy matrix of the equation (3.9) will transform into the monodromy matrix of the equation (4.2), which satisfy the expression (4.5).

Taking into account the differential relationships (4.2), (4.3) and (4.5), one can recalculate the Poisson bracket (3.22) by means of the identification

$$\bar{T}(\bar{l})(z; \lambda) = \bar{T}(z; \lambda) \quad (4.6)$$

for arbitrary  $z \in \mathbb{S}^1$  and  $\lambda \in \mathbb{C}^1$ . It entails the following tensor expression for the reduced phase space  $\bar{M}_\xi$ :

$$\begin{aligned} & \{ \bar{T}(\bar{l})(z; \lambda) \otimes \bar{T}(\bar{l})(z; \mu) \}_\xi = \\ &= \int_z^{z+2\pi} dx \int_z^{z+2\pi} dy \{ F(z + 2\pi, x; \lambda) \bar{l}(x; \lambda) F(x, z; \lambda) \otimes F(z + 2\pi, y; \mu) \bar{l}(y; \mu) F(y, z; \mu) \}_\xi = \\ &= \int_z^{z+2\pi} dx \int_z^{z+2\pi} dy \{ (F(z + 2\pi, x; \lambda) \otimes \mathbb{I})(\bar{l}(x; \lambda) \otimes \mathbb{I})(F(x, z; \lambda) \otimes \mathbb{I}, \\ & \quad \mathbb{I} \otimes F(z + 2\pi, y; \mu)(\mathbb{I} \otimes \bar{l}(y; \mu))(\mathbb{I} \otimes F(y, z; \mu)) \}_\xi = \\ &= \int_z^{z+2\pi} dx \int_z^{z+2\pi} dy F(z + 2\pi, x; \lambda) \otimes F(z + 2\pi, y; \mu) \{ \bar{l}(x; \lambda) \otimes \bar{l}(y; \mu) \}_\xi F(x, z; \lambda) \otimes F(y, z; \mu) \\ & \quad \int_z^{z+2\pi} dx \int_z^{z+2\pi} dy F(z + 2\pi, x; \lambda) \otimes F(z + 2\pi, y; \mu) \bar{\omega}(\lambda, \mu; x, y) F(x, z; \lambda) \otimes F(y, z; \mu), \end{aligned} \quad (4.7)$$

where  $z \in \mathbb{S}^1$ ,  $\lambda, \mu \in \mathbb{C}$  and, by definition, we put

$$\{ \bar{l}(x; \lambda) \otimes \bar{l}(y; \mu) \}_\xi := \bar{\omega}(\lambda, \mu; x, y) = \sum_{i,k=0}^N \bar{\omega}_{ik}(\lambda, \mu; x, y) \partial_x^i \partial_y^k \delta(x - y) \quad (4.8)$$

with local functional matrices  $\bar{\omega}_{ik}(\lambda, \mu; x, y) \in \tilde{\mathcal{G}}^* \otimes \tilde{\mathcal{G}}^*$ , satisfying the antisymmetry property:

$$P\bar{\omega}_{ik}(\lambda, \mu; x, y)P = -\bar{\omega}_{ki}(\mu, \lambda; x, y) \quad (4.9)$$

for all  $i, k = \overline{1, N}$ ,  $x, y \in \mathbb{S}^1$ ,  $\lambda, \mu \in \mathbb{C}$  and the permutation operator  $P : \tilde{\mathcal{G}}^* \otimes \tilde{\mathcal{G}}^*$ , acting as  $PA \otimes BP := B \otimes A$  for any  $A, B \in \tilde{\mathcal{G}}^*$ . Similarly to the calculation from [8, 18, 17] one obtains from (4.8) that

$$\{\bar{T}(z; \lambda) \otimes \bar{T}(z; \mu)\}_{\xi} = \int_z^{z+2\pi} dx \bar{F}(z+2\pi, x; \lambda) \otimes \bar{F}(z+2\pi, x; \mu) \bar{\Omega}(\lambda, \mu; x) \bar{F}(x, z; \lambda) \otimes \bar{F}(x, z; \mu), \quad (4.10)$$

where the matrix  $\bar{\Omega}(\lambda, \mu; x) \in \tilde{\mathcal{G}}^* \otimes \tilde{\mathcal{G}}^*$  for all  $\lambda, \mu \in \mathbb{C}, x \in \mathbb{S}^1$ , depends only from element  $\bar{l} \in \tilde{\mathcal{G}}^*$ .

The expression (4.10) allows a very compact representation

$$\{\bar{T}(z; \lambda) \otimes \bar{T}(z; \mu)\}_{\xi} = \mathcal{R}(\lambda, \mu; z) \bar{T}(z; \lambda) \otimes \bar{T}(z; \mu) - \bar{T}(z; \lambda) \otimes \bar{T}(z; \mu) \mathcal{R}(\lambda, \mu; z), \quad (4.11)$$

if the tensor  $\mathcal{R}$ -matrix  $\mathcal{R} \in \tilde{\mathcal{G}} \otimes \tilde{\mathcal{G}}^*$  satisfies for  $z \in \mathbb{S}^1$  and  $\lambda, \mu \in \mathbb{C}$  the differential relationship

$$\frac{d}{dx} \mathcal{R}(\lambda, \mu; z) + [\mathcal{R}(\lambda, \mu; z), l(z; \lambda) \otimes \mathbb{I} + \mathbb{I} \otimes l(z; \mu)] = \Omega(\lambda, \mu; z). \quad (4.12)$$

If to define the mapping  $R : \tilde{\mathcal{G}} \rightarrow \tilde{\mathcal{G}}$  as

$$R(Y) := \text{res}_{\mu=0} \int_0^{2\pi} dy \mathcal{R}(\lambda, \mu; y) \delta(x-y) Y(y; \mu) \quad (4.13)$$

for any  $Y \in \tilde{\mathcal{G}}^*$ , then the relationship (4.12) can be easily presented in the following operator form:

$$-(X, \frac{dR}{dx}(Y))_{-1} + (\bar{l}, [X, Y]_R) = (X, R(Y))_{-1}, \quad (4.14)$$

which holds for any  $X, Y \in \tilde{\mathcal{G}}$ , where we denoted

$$[X, Y]_R := [-R^*(x), Y] + [X, R(Y)]. \quad (4.15)$$

The result (4.14) can be used for rewriting the Poisson bracket (4.11) as

$$\begin{aligned} & \{(X, \bar{T}(\bar{l}))_{-1}, (Y, \bar{T}(\bar{l}))_{-1}\}_{\xi} = \\ &= (\bar{l}, [\bar{F}X\bar{F}_{2\pi}, \bar{F}Y\bar{F}_{2\pi}]_R)_{-1} - \left( \bar{F}X\bar{F}_{2\pi}, \frac{R}{dx}(\bar{F}Y\bar{F}_{2\pi}) \right)_{-1} = \\ &= (\bar{l}, [\nabla(X, \bar{T})(\bar{l}), \nabla(Y, \bar{T})(\bar{l})]_R)_{-1} - \left( \nabla(X, \bar{T})(\bar{l}), \frac{d}{dx}R(\nabla(Y, \bar{T})(\bar{l})) \right)_{-1} - \\ & - \left( R^*(\nabla(X, \bar{T})(\bar{l})), \frac{d}{dx}(\nabla(Y, \bar{T})(\bar{l})) \right)_{-1}, \end{aligned} \quad (4.16)$$

where  $\bar{F} := \bar{F}(\bar{l})(x, y; \lambda)$ ,  $\bar{F}_{2\pi} := \bar{F}(\bar{l})(y+2\pi, x; \lambda) \in \tilde{\mathcal{G}}$ ,  $x, y \in \mathbb{S}^1, \lambda \in \mathbb{C}$ , and we defined the gradients  $\nabla(X, \bar{T})(\bar{l})$  and  $\nabla(Y, \bar{T})(\bar{l}) \in \tilde{\mathcal{G}}$  by means of the standard definition

$$(\nabla f(\bar{l}), Z)_{-1} := \frac{d}{d\varepsilon} f(\bar{l} + \varepsilon Z) \Big|_{\varepsilon=0} \quad (4.17)$$

for any smooth functional  $f \in \mathcal{D}(\tilde{\mathcal{G}}^*)$  and arbitrary  $Z \in \tilde{\mathcal{G}}^*$ .

It is easy to observe that under the antisymmetry condition  $R^* = -R$  the righthand side of (4.16) equals exactly the Lie-Poisson bracket [8, 15, 13, 9, 6] for the functionals  $(X, \bar{T})$  and  $(Y, \bar{T}) \in \mathcal{D}(\tilde{\mathcal{G}}^*)$  on the adjoint space  $\tilde{\mathcal{G}}^* = \tilde{\mathcal{G}}^* \oplus \mathbb{C}$  with respect to a new commutator structure  $[\cdot, \cdot]_R$  on the centrally extended Lie algebra  $\hat{\mathcal{G}}$ : for any  $(X, c), (Y, r) \in \hat{\mathcal{G}}$  the commutator

$$[(X, c), (Y, r)]_R := \left( [X, Y]_R, \left( \frac{d}{dx} X, R(Y) \right)_{-1} + \left( \frac{d}{dx} R(X), Y \right)_{-1} \right) \quad (4.18)$$



where the classical  $R$ -structure on the Lie algebra  $\tilde{\mathcal{G}} [X, Y]_R := [R(x), Y] + [X, R(Y)]$  under some conditions on the mapping  $R : \tilde{\mathcal{G}} \rightarrow \tilde{\mathcal{G}}$  can generate on  $\tilde{\mathcal{G}}$  a new Lie structure (but, evidently, it not must!).

The obtained result we will formulate as the next statement.

**Proposition 4.1** *The Marsden-Weinstein reduced canonical Poisson structure on the phase space  $\bar{M}$  for the monodromy matrix  $\bar{T}(\bar{l}) \in \tilde{\mathcal{G}}$  exactly coincides with the corresponding classical Lie-Poisson AKS-bracket on the centrally extended basis Lie algebra  $\tilde{\mathcal{G}}$  subject to the  $R$ -structure (4.18) in case of its antisymmetry.*

If the antisymmetry property for the mapping  $R : \tilde{\mathcal{G}} \rightarrow \tilde{\mathcal{G}}$  does not hold, the respectively generated Lie-Poisson type bracket on the functional space  $\mathcal{D}(\tilde{\mathcal{G}}^*)$  can be, owing to (4.16), defined as follows: for any  $f, g \in \mathcal{D}(\tilde{\mathcal{G}}^*)$  the bracket

$$\{f(\bar{l}), g(\bar{l})\}_\xi := (\bar{l}, [\nabla f(\bar{l}), \nabla g(\bar{l})]_R)_{-1} + \left( \frac{d}{dx} \nabla f(\bar{l}), R(\nabla g(\bar{l})) \right)_{-1} + \left( \frac{d}{dx} (R \nabla f(\bar{l})), \nabla g(\bar{l}) \right)_{-1} \quad (4.19)$$

where the generalized  $R$ -structure  $[\cdot, \cdot]_R$  on  $\tilde{\mathcal{G}}$  is given by the expression (4.15).

## 5 $D$ -structure and the generalized $R$ -structure relationship analysis.

As it was stated above, the reduced on the phase space  $\bar{M}_\xi$  Poisson bracket

$$\{(X, \bar{T}), (Y, \bar{T})\}_\xi = (\bar{T}, [X, Y]_D)_{-1}, \quad (5.1)$$

where for any  $X, Y \in \tilde{\mathcal{G}}$  the corresponding  $D$ -structure on the Lie algebra  $\tilde{\mathcal{G}}$  is defined by the classical expression (3.23) and the mapping (3.24). It is natural to assume that there exists a relationship between the  $D$ -structure  $D : \tilde{\mathcal{G}} \rightarrow \tilde{\mathcal{G}}$  and the  $R$ -structure  $R : \tilde{\mathcal{G}} \rightarrow \tilde{\mathcal{G}}$ , described above in Section 3.

Assume, for brevity, that the  $R$ -structure (4.13) is antisymmetric, that is  $R^* = -R$ . Then it is easy to check that the following algebraic relationship

$$D(X) := \frac{1}{2} R(\bar{T}X + X\bar{T}) \quad (5.2)$$

holds for any  $X \in \tilde{\mathcal{G}}$ . Really, the expression (4.11) is equivalent to the following

$$\{(\bar{T}, X), (\bar{T}, Y)\}_\xi = (\bar{T}X, R(\bar{T}Y))_{-1} - (X\bar{T}, R(Y\bar{T}))_{-1}. \quad (5.3)$$

Now, substituting the expression (5.2) into (3.22), one obtains that

$$\begin{aligned} & \{(\bar{T}, X), (\bar{T}, Y)\}_\xi = \\ &= \frac{1}{2} (\bar{T}, [R(\bar{T}X + X\bar{T}), Y] + [X, R(\bar{T}Y + Y\bar{T})])_{-1} = \\ &= \frac{1}{2} ([Y, \bar{T}], R(\bar{T}X))_{-1} + \frac{1}{2} ([Y, \bar{T}], R(X\bar{T}))_{-1} + \\ &+ \frac{1}{2} ([\bar{T}, X], R(\bar{T}Y))_{-1} + \frac{1}{2} ([\bar{T}, X], R(Y\bar{T}))_{-1} = \\ &= \frac{1}{2} (Y\bar{T}, R(\bar{T}X))_{-1} - \frac{1}{2} (\bar{T}Y, R(\bar{T}X))_{-1} + \\ &+ \frac{1}{2} (Y\bar{T}, R(X\bar{T}))_{-1} - \frac{1}{2} (\bar{T}Y, R(X\bar{T}))_{-1} + \\ &+ \frac{1}{2} (\bar{T}X, R(\bar{T}Y))_{-1} - \frac{1}{2} (X\bar{T}, R(\bar{T}X))_{-1} + \\ &+ \frac{1}{2} (\bar{T}X, R(Y\bar{T}))_{-1} - \frac{1}{2} (X\bar{T}, R(Y\bar{T}))_{-1} = \\ &= (\bar{T}X, R(\bar{T}Y))_{-1} - (X\bar{T}, R(Y\bar{T}))_{-1}, \end{aligned} \quad (5.4)$$

which coincides exactly with (5.3).

Rewrite now for convenience the operator relationship (4.1) in the tensor form as

$$(\bar{l} \otimes \mathbb{I})D - D(\mathbb{I} \otimes \bar{l}) - D_x = \mathbb{I} - \bar{T}(\bar{l}) \otimes \bar{Q}(\bar{l}), \quad (5.5)$$

where the tensor  $D \in \tilde{\mathcal{G}} \otimes \tilde{\mathcal{G}}^*$ , owing to the action (5.2), equals

$$D = \frac{1}{2}(R(\mathbb{I} \otimes \bar{T}) + (\mathbb{I} \otimes \bar{T})R). \quad (5.6)$$

Substituting the expression (5.6) into the equation (5.5) and taking into account the determining equation (4.12)

$$[\bar{l} \otimes \mathbb{I} + \mathbb{I} \otimes \bar{l}, R] - \frac{d}{dx}R = \bar{\Omega}, \quad (5.7)$$

one obtains the relationship for tensors  $\bar{Q}$  and  $\bar{\Omega} \in \tilde{\mathcal{G}} \rightarrow \tilde{\mathcal{G}}^*$ :

$$2\mathbb{I} \otimes \mathbb{I} - 2\bar{Q} - \bar{\Omega} = [R, \mathbb{I} \otimes \bar{T}\bar{l}] + (\mathbb{I} \otimes \bar{T})R(\mathbb{I} \otimes \bar{l}) - (\mathbb{I} \otimes \bar{l})R(\mathbb{I} \otimes \bar{T}). \quad (5.8)$$

The latter makes two  $R$ - and  $D$ -structures on the Lie algebra  $\tilde{\mathcal{G}}$  to be compatible. Remark also that the  $D$ -structure (5.2) is not antisymmetric despite the  $R$ -structure was assumed to be antisymmetric. Concerning the  $D$ -structure determining equation (5.5) one can anticipate that a study of its solutions would describe a set of nonlinear dynamical systems on the reduced phase space  $\bar{M}_\xi$  possessing a priori an infinite hierarchy of commuting to each other conservation laws.

## 6 Conclusion

We have considered the standard canonically symplectic phase space  $M := T^*(\tilde{\mathcal{G}})$ , generated by the centrally extended basis manifold to be an affine loop Lie algebra  $\tilde{\mathcal{G}}$  on the circle  $\mathbb{S}^1$ . Subject to the standard Hamiltonian Lie algebra  $\tilde{\mathcal{G}}$ -action on  $M$ , with respect which the symplectic structure on  $M$  is invariant, constructed the corresponding momentum mapping and carried out the standard Marsden-Weinstein reduction of the manifold  $M$  upon the reduced phase space  $\bar{M}_\xi$  endowed with the reduced Poisson bracket  $\{\cdot, \cdot\}_\xi$ . The latter allows to construct on the phase space  $\bar{M}_\xi$  commuting to each other vector fields which are equivalent to some nonlinear dynamical systems possessing an infinite hierarchy of commuting conservation laws. Moreover, these mentioned commuting vector fields on  $\bar{M}_\xi$  realize exactly their corresponding Lax type representations.

Presented detailed analysis of commutation properties for the related flows on the basis manifold makes it possible to define a suitable  $D$ -structure on the Lie algebra  $\tilde{\mathcal{G}}$ , deeply related with the corresponding classical  $R$ -structure on  $\tilde{\mathcal{G}}$ , generated by the reduced Poisson bracket on the phase space  $\bar{M}_\xi$ . As a bi-product of our analysis we stated that these  $R$ - and  $D$ -structures are completely equivalent to a suitably generalized classical Lie-Poisson-Adler-Kostant-Symes-Kirillov-Berezin structure on the adjoint space  $\tilde{\mathcal{G}}^*$ . We derived also the determining equation for the  $D$ -structure, classifying the generalized Lax type integrable nonlinear dynamical systems on the reduced phase space  $\bar{M}_\xi$ , whose respectively defined  $R$ -structures are not necessary both antisymmetric and local, as it was before described in [3, 4] by means of an other approach. It is worth also to mention that the reduction scheme devised in this work can be applied also to the centrally extended algebra of pseudo-differential operators and affine loop algebras on the circle  $\mathbb{S}^1$ .

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